

The two-jet of the curvature tensor of an Einstein manifold

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Abstract

The two-jet $(R_p \nabla R_p, \nabla^2 R_p)$ of the curvature tensor at some point p of a pseudo-Riemannian manifold is called Einstein if the Ricci tensor is a multiple of the metric tensor g_p and additionally its first two covariant derivatives vanish at p . Following the Jet Isomorphism Theorem of pseudo-Riemannian geometry, we derive necessary and sufficient conditions for the Einstein property in terms of the symmetrization of $(R_p, \nabla R_p, \nabla^2 R_p)$ (i.e. in terms of the Jacobi operator and its first two covariant derivatives along arbitrary geodesics emanating from p). A central role is played by the Weitzenböck formula for the Laplacian $d^\nabla \delta^\nabla + \delta^\nabla d^\nabla$ acting on sections of the vector bundle of algebraic curvature tensors. As an application, we study linear Jacobi relations of order two on Einstein manifolds.

1 Introduction

Let $(M, \langle \cdot, \cdot \rangle)$ be a pseudo-Riemannian manifold and $p \in M$. We denote by ∇ and R the Levi-Civita connection and the curvature tensor. It was shown in [9] that the condition $\nabla^k R = 0$ already implies that M is locally symmetric. In order to find less restrictive conditions on the curvature tensor, recall that the Jacobi operator along some geodesic γ is defined by $\mathcal{R}_\gamma(t; x, y) := R(x, \dot{\gamma}(t), \dot{\gamma}(t), y)$ for all $t \in \mathbb{R}$ and $x, y \in T_{\gamma(t)} M$. Further, let $\text{Sym}^k V$ denote the k -th symmetric power of an arbitrary vector space V . Thus the Jacobi operator is a section of $\text{Sym}^2 TM^*$ along γ .

Furthermore, let $\nabla^k R$ denote the k -th covariant derivative of the curvature tensor. Then the k -th covariant derivative of \mathcal{R}_γ is given by

$$\mathcal{R}_\gamma^{(k)}(x, y) := \frac{\nabla^k}{dt^k} \mathcal{R}_\gamma(x, y) = \nabla_{\dot{\gamma}, \dots, \dot{\gamma}}^{(k)} R(\dot{\gamma}, x, y, \dot{\gamma}) \quad (1)$$

which is a section of $\text{Sym}^2 TM^*$ along γ again. However, because of (1), we can define the k -th covariant derivative of the Jacobi operator without reference to a special geodesic as follows:

Definition 1. Let $\mathcal{R}^{(k)}$ denote the section of $\text{Sym}^{k+2} TM^* \otimes \text{Sym}^2 TM^*$ which is uniquely characterized by

$$\mathcal{R}^{(k)}(\underbrace{\xi, \dots, \xi}_{k+2}; x, y) := \nabla_{\xi, \dots, \xi}^{(k)} R(x, \xi, \xi, y) \quad (2)$$

for all $p \in M$ and $\xi, x, y \in T_p M$ via the polarization formula. We will call $\mathcal{R}^{(k)}$ the symmetrized k -th covariant derivative of the curvature tensor.

1.1 The Einstein condition on the symmetrized two-jet of the curvature tensor

Since we are interested in local properties of the metric, from now on we assume that M is a vector space V and p is the origin. Thus $\langle \cdot, \cdot \rangle := g|_0$ equips V with the structure of a pseudo-euclidean vector space. The k -jet associated with g is the collection $(R|_0, \nabla R|_0, \dots, \nabla^k R|_0)$ obtained by evaluating the curvature tensor and its first k covariant derivatives at the origin. A k -jet can also be defined in a purely algebraic manner, see Def. 4. In the following an (algebraic) k -jet will be denoted by $(R, \nabla R, \dots, \nabla^k R)$ (the symbol $|_0$ for evaluation at the origin gets dropped only in order to keep the notation more easy.)

Definition 2. We say that an algebraic k -jet $(R, \nabla R, \dots, \nabla^k R)$ is Einstein if the Ricci tensor is a multiple of $\langle \cdot, \cdot \rangle$ and the vanishing conditions $\nabla \text{ric} = 0, \dots, \nabla^k \text{ric} = 0$ together hold.

It is known that $\nabla \text{ric} = 0$ implies that ∇R is totally trace-free, see Lemma 4.

Let $\alpha \in \text{Sym}^k V^*$ and $\beta \in \text{Sym}^\ell V^*$. We define the symmetric product $\alpha \odot \beta \in \text{Sym}^{k+\ell} V^*$ as the product of the corresponding polynomial functions, i.e.

$$\alpha \odot \beta(\xi, \dots, \xi) := \alpha(\xi)\beta(\xi) . \quad (3)$$

Theorem 1. *An algebraic two-jet $(R, \nabla R, \nabla^2 R)$ is Einstein if and only if $\text{ric} \in \mathbb{R} \langle \cdot, \cdot \rangle$, $\nabla \text{ric} = 0$ and*

$$\mathcal{R}^2 - \frac{1}{n+4} \mathcal{R} * \mathcal{R} \odot \langle \cdot, \cdot \rangle . \quad (4)$$

*is totally trace-free. Here $\mathcal{R} * \mathcal{R}$ denotes the algebraic Jacobi operator associated with the algebraic curvature tensor $R * R$ (the latter will be defined in (19) below.)*

1.2 Linear Jacobi relations

Definition 3. A linear Jacobi relation of order k is a dependency relation

$$\mathcal{R}^{k+1} = c_{k-1} \langle \cdot, \cdot \rangle \odot \mathcal{R}^{k-1} + c_{k-3} \langle \cdot, \cdot \rangle \odot \langle \cdot, \cdot \rangle \odot \mathcal{R}^{k-3} + \dots \quad (5)$$

in the space of sections of the vector bundle $\text{Sym}^{k+3} TM^* \otimes \text{Sym}^2 TM^*$.

In other words, this means that there exist constants c_{k-1}, c_{k-3}, \dots such that

$$\mathcal{R}_\gamma^{k+1} = c_{k-1} \langle \dot{\gamma}, \dot{\gamma} \rangle \mathcal{R}_\gamma^{k-1} + c_{k-3} \langle \dot{\gamma}, \dot{\gamma} \rangle^2 \mathcal{R}_\gamma^{k-3} + \dots . \quad (6)$$

for all geodesics γ .

For example, a linear Jacobi relation of order zero means that $\nabla R = 0$, i.e. M is locally symmetric. Further, examples with (minimal) linear Jacobi relations of order two and four are known, see [1, 8]. Among them there are certain homogeneous Einstein spaces. On the other hand it is an open question whether there exist pseudo-Riemannian spaces with a (minimal) linear Jacobi relation of uneven order at all. Using Theorem 1 in combination with (36) below, we will prove the following result on linear Jacobi relations of order one:

Corollary 1. *Suppose for a pseudo-Riemannian Einstein manifold there exists a constant c such that*

$$\mathcal{R}_\gamma^2 = c \langle \dot{\gamma}, \dot{\gamma} \rangle \mathcal{R}_\gamma^0 \quad (7)$$

for all geodesics γ . Then the curvature tensor of M satisfies

$$\nabla^* \nabla R = -\frac{(n+4)c}{2} R. \quad (8)$$

In particular, the constant c is negative on a compact Riemannian manifold.

2 Symmetrized/anti-symmetrized covariant derivatives of the curvature tensor

Instead of working directly with the symmetrization of the covariant derivatives of the curvature tensor defined in (2), there is no flaw in passing to the symmetrized/anti-symmetrized covariant derivative, since this contains exactly the same information. In fact, this has even some advantages for the calculations. The construction is as follows:

Consider the Young diagram of shape $(k+2, 2)$ given by

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 5 & \cdots & k+4 \\ \hline 2 & 4 & & & \\ \hline \end{array} \quad (9)$$

for any $k \geq 0$. The corresponding Young symmetrizer defines an endomorphism on covariant tensors of degree $k+4$ which can roughly be described as a specific symmetrization/anti-symmetrization operation (see [4, Ch. 6]). The *Schur functor* associated with (9) assigns to each vector space V the image $\mathcal{C}_k(V)$ of this endomorphism in $\bigotimes^{k+4} V^*$. The complexification $\mathcal{C}_k(V) \otimes \mathbb{C}$ is known to be an irreducible representation of $\mathfrak{sl}(V, \mathbb{C})$ of highest weight $(k+2)L_1 + 2L_2$ in the notation from [4], called Weyls construction. Once the vector space V is kept fix, we write \mathcal{C}_k instead of $\mathcal{C}_k(V)$. For a proof of the following Lemma see for example [5, Sec. 3]:

Lemma 1. *The linear spaces $\mathcal{C}_k \subset \bigotimes^{k+4} V^*$ can also be described as follows:*

$$\mathcal{C}_0 := \{R \in \text{Sym}^2(\Lambda^2 V^*) \mid R \text{ satisfies the first Bianchi-identity}\}, \quad (10)$$

$$\mathcal{C}_1 := \{\nabla R \in V^* \otimes \mathcal{C}_0 \mid \nabla R \text{ satisfies the second Bianchi identity}\}, \quad (11)$$

$$\mathcal{C}_k := \text{Sym}^k V^* \otimes \mathcal{C}_0 \cap \text{Sym}^{k-1} V^* \otimes \mathcal{C}_1 \text{ for } k \geq 2. \quad (12)$$

Thus \mathcal{C}_0 is and \mathcal{C}_1 are the spaces of (algebraic) curvature tensors and their covariant derivatives, respectively. More generally, \mathcal{C}_k is the space of (algebraic) linear k -jets of curvature tensors on V , see also [7, Def. 4]. In fact, according to the Jet Isomorphism Theorem of pseudo-Riemannian geometry (see [7]), the prefix “algebraic” can actually be omitted: let some pseudo-Riemannian metric on the differentiable manifold V be given whose curvature tensor satisfies $R|_0 = 0, \dots, \nabla^{k-1} R|_0 = 0$ (where again $|_0$ denotes evaluation at the origin). Then $\nabla^k R|_0 \in \mathcal{C}_k$ by means of the Ricci identity. Further, any element of \mathcal{C}_k is obtained in this way.

The *Hook Length Formula* [4, 4.12] in combination with [4, Lemma 4.26] implies that

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 5 & \cdots & k+4 \\ \hline 2 & 4 & & & \\ \hline \end{array} \nabla_{x_4, \dots, x_{k+4}}^{(k)} R(x_1, x_2, x_3, x_4) = 2(k+3)(k+2)k! \nabla_{x_4, \dots, x_{k+4}}^{(k)} R(x_1, x_2, x_3, x_4) \quad (13)$$

for all $\nabla^k R \in \mathcal{C}_k$ and $k \geq 0$. For example, the factor $2(k+3)(k+2)k!$ occurring on the right hand side of (13) is 12 for $k = 0$, 24 for $k = 1$ and 80 for $k = 2$, respectively. It is not very difficult to check (13) from (10)-(12) via explicit calculations, see [7, Sec. 3]. In fact, in case of $R = 0$ the calculations given in Sec. 4.1 below reduce to the previously mentioned ones.

Example 1. Let $\nabla^{k-2\ell} R \in \mathcal{C}_{k-2\ell}$ and consider the tensor $\mathcal{R}^{k-2\ell} \odot \langle \cdot, \cdot \rangle^\ell \in \text{Sym}^{k+2} V^* \otimes \text{Sym}^2$, characterized by

$$\mathcal{R}^{k-2\ell} \odot \langle \cdot, \cdot \rangle^\ell(x, x, x, \dots, x; y, y) := \mathcal{R}^k(x; y, y) \langle x, x \rangle^\ell \text{ (see also (3))} . \quad (14)$$

These are the tensors which occur in (5). Further, recall that the Kulkarni-Nomizu product is the linear map

$$\oslash : \text{Sym}^{k+2} V^* \otimes \text{Sym}^2 V^* \rightarrow \mathcal{C}_k$$

given by

$$\oslash(h_1 \otimes h_2)(x_1, \dots, x_k; x_a, x_b, x_c, x_d) := \begin{cases} h_1(x_1, \dots, x_k, x_a, x_c) h_2(x_b, x_d) \\ -h_1(x_1, \dots, x_k, x_b, x_c) h_2(x_a, x_d) \\ -h_1(x_1, \dots, x_k, x_a, x_d) h_2(x_b, x_c) \\ +h_1(x_1, \dots, x_k, x_b, x_d) h_2(x_a, x_c) . \end{cases} \quad (15)$$

Then, by the very definition of the Young symmetrizer associated with (9)

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 5 & \cdots & k+4 \\ \hline 2 & 4 & & & \\ \hline \end{array} \nabla^{k-2\ell} R \otimes \langle \cdot, \cdot \rangle^\ell(x_1, \dots, x_{k+4}) = -2(k+2)! \oslash \mathcal{R}^{k-2\ell} \odot \langle \cdot, \cdot \rangle^\ell(x_1, \dots, x_{k+4}) . \quad (16)$$

Lemma 2. *A pseudo-Riemannian manifold satisfies a linear Jacobi relation (5) if and only if*

$$\begin{aligned} & \begin{array}{|c|c|c|c|} \hline 1 & 2 & \cdots & k+3 \\ \hline a & b & & \\ \hline \end{array} \nabla_{x_3, \dots, x_{k+3}}^{(k+1)} R(x_1, x_a, x_b, x_2) \\ &= a_{k-1} \begin{array}{|c|c|c|c|} \hline 1 & 2 & \cdots & k+3 \\ \hline a & b & & \\ \hline \end{array} \nabla_{x_3, \dots, x_{k+1}}^{(k-1)} R(x_1, x_a, x_b, x_2) \langle x_{k+2}, x_{k+3} \rangle \\ &+ a_{k-3} \begin{array}{|c|c|c|c|} \hline 1 & 2 & \cdots & k+3 \\ \hline a & b & & \\ \hline \end{array} \nabla_{x_3, \dots, x_{k-1}}^{(k-3)} R(x_1, x_a, x_b, x_2) \langle x_k, x_{k+1} \rangle \langle x_{k+2}, x_{k+3} \rangle + \cdots . \end{aligned} \quad (17)$$

Proof. Set

$$\mathcal{N}_k := \{h \in \text{Sym}^k V^* \otimes \text{Sym}^2 V^* \mid \forall \xi \in V : h(\underbrace{\xi, \dots, \xi}_{k+1}, \cdot) = 0\} \quad (18)$$

Because of the first Bianchi identity, $\mathcal{R}^{k-2\ell} \odot \langle \cdot, \cdot \rangle^\ell \in \mathcal{N}_{k+2}$. Furthermore, the Kulkarni-Nomizu product $\oslash : \mathcal{N}_{k+2} \rightarrow \mathcal{C}_k$ is a non-zero multiple of the linearization of some isomorphism of affine jet-spaces appearing in the Jet Isomorphism Theorem of pseudo-Riemannian geometry, see [7, Eq. (48)]. In particular, the Kulkarni-Nomizu product has a trivial kernel on \mathcal{N}_{k+2} for any $k \geq 0$. The assertion follows from (16). \square

3 Some identities for algebraic curvature tensors

For simplicity, we restrict our considerations to the Riemannian case. The generalization to the pseudo-Riemannian case is straightforward. Let V be a Euclidean vector space and $\{e_1, \dots, e_n\}$ be an orthonormal basis. Given a curvature tensor and some $A \in \bigotimes^k V^*$ we define

$$R * A(x_1, \dots, x_k) := - \sum_{i=1}^k \sum_{j=1}^n R_{x_i, e_j} \cdot A(x_1, \dots, \overbrace{e_j}^i, \dots, x_k) \quad (19)$$

Here $R_{x,y} \in \mathfrak{so}(V)$ means the curvature endomorphism $R(x, y, \cdot)$ for all $x, y \in V$. Further, the dot \cdot means the usual action of skew-adjoint endomorphisms on arbitrary tensors through algebraic derivation, i.e.

$$-B \cdot A(x_1, \dots, x_k) = \sum_{i=1}^k A(x_1, \dots, B x_i, \dots, x_k)$$

for any endomorphism A and $B \in \bigotimes^k V^*$. By definition, the tensor $R * A$ has the same symmetries as A . In particular, if A is a curvature tensor, then $R * A$ is a curvature tensor again.

Remark 1. A straightforward calculation shows that $R * A$ is the zeroth order term Γ which appears in the definition of the Lichnerowicz Laplacian on covariant tensor fields of valence k , see [2, 1.143]. In [10] the operator $R*$ is denoted by $q(R)$, see Formula (3.9) there. For example, $R * \alpha(x) = \alpha(\text{Ric}(x))$ for every 1-form α , e.g. $R * = n \text{Id}$ on the vector bundle of 1-forms on the n -dimensional round sphere of radius one.

Lemma 3. *Let two algebraic curvature tensors R and R' be given. Then:*

$$R * R'(x_1, x_2, x_3, x_4) = - \sum_{i=1}^n \left\{ \begin{array}{l} 2 R_{x_1, e_i} \cdot R'(e_i, x_2, x_3, x_4) - 2 R_{x_2, e_i} \cdot R'(e_i, x_1, x_3, x_4) \\ + R'_{x_2, x_4} \cdot \text{ric}(x_1, x_3) - R'_{x_2, x_3} \cdot \text{ric}(x_1, x_4) \\ + R'_{x_1, x_3} \cdot \text{ric}(x_2, x_4) - R'_{x_1, x_4} \cdot \text{ric}(x_2, x_3) \end{array} \right. \quad (20)$$

$$R * R'(x, y, x, y) = -2 R'_{x, y} \cdot \text{ric}(x, y) - 4 \sum_{i=1}^n R_{x, e_i} \cdot R'(e_i, y, x, y), \quad (21)$$

$$\sum_{i=1}^n R * R'(x, e_i, y, e_i) = -R * \text{ric}'(x, y), \quad (22)$$

$$\sum_{i=1}^n R * \text{ric}'(e_i, e_i) = 0. \quad (23)$$

Proof. For (20): Since R' has the algebraic properties of a curvature tensor it follows immediately from (19) that

$$R * R'(x, y, x, y) = -2 \sum_{i=1}^n R_{x, e_i} \cdot R'(e_i, y, x, y) + R_{y, e_i} \cdot R'(e_i, x, y, x). \quad (24)$$

Further, let us denote r.h.s. of this equation by $\tilde{R}(x_1, x_2, x_3, x_4)$. Thus it is easy to see that $\tilde{R}(x, y, y, x) = R * R'(x, y, y, x)$. Further, a straightforward calculation (using the first Bianchi identity for R') shows that $\tilde{R}(x_1, x_2, x_3, x_4) = \tilde{R}(x_3, x_4, x_1, x_2)$ (pair symmetry). Thus (24) already implies (20).

For (21): we have

$$R_{x, e_i} \cdot R'(e_i, y, y, x) + R'(\text{Ric}(x), y, y, x) - \langle R(e_i, x, x, \cdot), R'(e_i, y, y) \rangle = \quad (25)$$

$$- \sum_{i=1}^n \underbrace{R'(e_i, R(x, e_i, y), y, x)}_{=\langle R'(x, y, e_i), R(e_i, x, y) \rangle} + \underbrace{R'(e_i, y, R(x, e_i, y), x)}_{=\langle R'(e_i, y, x), R(e_i, x, y) \rangle} \stackrel{1. \text{ Bianchi}}{=} \quad (26)$$

$$- 2 \sum_{i=1}^n \langle R(e_i, x, y), R'(e_i, y, x) \rangle + \sum_{i=1}^n \langle R(e_i, x, y), R'(e_i, x, y) \rangle. \quad (27)$$

Further, recall the formula

$$\sum_{i=1}^n \langle A e_i, B e_i \rangle = \text{tr}(B^* \circ A) = \text{tr}(A \circ B^*) = \sum_{i=1}^n \langle A^* e_i, B^* e_i \rangle$$

for any pair of endomorphisms (A, B) on V . Here A^* means the transpose endomorphism. Applying this formula to $A(u) := R(u, x, y)$ and $B(u) := R'(u, y, x)$ and using that $A^*(u) = R(u, y, x)$ and $B^*(u) = R'(u, x, y)$, we see that both $\langle R(e_i, x, y), R'(e_i, y, x) \rangle$ and $\langle R(e_i, x, y), R'(e_i, x, y) \rangle$ are symmetric in $\{x, y\}$. Hence,

$$\sum_{i=1}^n R_{x, e_i} \cdot R(e_i, y, y, x) - R_{y, e_i} \cdot R(e_i, x, x, y) = \underbrace{R'(y, x, x, \text{Ric}(y)) - R'(x, y, y, \text{Ric}(x))}_{=2 R'_{x, y} \cdot \text{ric}(x, y)} . \quad (28)$$

Therefore, by means of (24),

$$R * R'(x, y, y, x) = 2 R'_{x, y} \cdot \text{ric}(x, y) - 4 \sum_{i=1}^n R_{x, e_i} \cdot R'(e_i, y, y, x)$$

Now (24) follows.

For (22), we can suppose that $x = y$. Then the equation immediately follows from (24) and the fact that the trace commutes with the action of a skew-symmetric endomorphism. For (23), we have

$$\sum_{i, j} R_{e_i, e_j} \cdot \text{ric}(e_j, e_i) = - \sum_{i, j} R_{e_j, e_i} \cdot \text{ric}(e_j, e_i) = - \sum_{i, j} R_{e_i, e_j} \cdot \text{ric}(e_j, e_i) .$$

The result follows. \square

Recall that an algebraic curvature tensor R on a Euclidean vector space $(V, \langle \cdot, \cdot \rangle)$ is called Einstein if $\text{ric} = c \langle \cdot, \cdot \rangle$ for some $c \in \mathbb{R}$.

Corollary 2. *Suppose R is Einstein. Then*

$$R * R(x, y, y, x) = -4 \sum_{i=1}^n R_{x, e_i} \cdot R(e_i, y, y, x) = -4 \sum_{i=1}^n R_{y, e_i} \cdot R(e_i, x, x, y) . \quad (29)$$

Further, $R * R$ has vanishing Ricci tensor.

3.1 The Weitzenböck formula for the Laplacian on the vector bundle of algebraic curvature tensors

In [3, Prop. 4.2] there was shown a “Weitzenböck formula” for the Laplacian $d^\nabla \delta^\nabla + \delta^\nabla d^\nabla$ acting on sections of $\text{Hom}(\Lambda^2 T^* M, \Lambda^2 T^* M)$, the vector bundle of endomorphisms of the second exterior power of the cotangent bundle. In Proposition 1 and Corollary 3 below we will rewrite this formula for sections of the subbundle $\mathcal{C}_0(TM) \subset \text{Hom}(\Lambda^2 T^* M, \Lambda^2 T^* M)$, the vector bundle of algebraic curvature tensors, and specifically for the curvature tensor itself. This will be the cornerstone for our further calculations.

Definition 4. Let $(V, \langle \cdot, \cdot \rangle)$ be a pseudo Euclidean vector space.

(a) A triple $(R', \nabla R', \nabla^2 R') \in \mathcal{C}_0 \oplus (V^* \otimes \mathcal{C}_0) \oplus (V^* \otimes V^* \otimes \mathcal{C}_0)$ is briefly called an (algebraic) two-jet if

$$\text{the Ricci-identity: } \forall x, y \in V : \nabla_{x, y}^2 R - \nabla_{y, x}^2 R = R_{x, y} \cdot R$$

and

$$\text{the second Bianchi identity: } \nabla R \in \mathcal{C}_1, \nabla^2 R \in V^* \otimes \mathcal{C}_1$$

together hold, see (11).

- (b) More generally, given some $R \in \mathcal{C}_0$ (see (10)), a triple $(R', \nabla R', \nabla^2 R') \in \mathcal{C}_0 \oplus (V^* \otimes \mathcal{C}_0) \oplus (V^* \otimes V^* \otimes \mathcal{C}_0)$ is called the (algebraic) two-jet of a section of the vector bundle of algebraic curvature tensors if the Ricci-identity $\nabla_{x,y}^2 R' - \nabla_{y,x}^2 R' = R_{x,y} \cdot R'$ holds for all $x, y \in V$.

According to the Jet Isomorphism Theorem, a triple $(R, \nabla R, \nabla^2 R)$ is an algebraic two-jet if and only if there exists some pseudo-Riemannian metric on V with $g_0 = \langle \cdot, \cdot \rangle$ and such that $(R, \nabla R, \nabla^2 R)$ is the two-jet of the curvature tensor evaluated at the origin, see Sec. 1.1. Similarly, an algebraic curvature tensor R together with a triple $(R', \nabla R', \nabla^2 R')$ defines the algebraic two-jet of a section of the vector bundle of algebraic curvature tensors if and only if there exists some pseudo-Riemannian metric with $g_0 = \langle \cdot, \cdot \rangle$ and whose curvature tensor at the origin is R such that $(R', \nabla R', \nabla^2 R')$ is the two-jet of a section of the vector bundle of algebraic curvature tensors evaluated at the origin.

Further, recall that the divergence of R and the exterior derivative of the Ricci tensor are defined as follows:

$$\delta_x^\nabla R(y, z) := - \sum_{i=1}^n \nabla_{e_i} R(e_i, x, y, z), \quad (30)$$

$$d^\nabla \text{ric}(x, y, z) := \nabla_x \text{ric}(y, z) - \nabla_y \text{ric}(x, z) \quad (31)$$

for all $x, y, z \in V$. Then the second Bianchi identity implies

$$\delta_z^\nabla R(x, y) = d^\nabla \text{ric}(x, y, z). \quad (32)$$

Next we state the Weitzenböck formula for the Laplacian $d^\nabla \delta^\nabla + \delta^\nabla d^\nabla$ acting on the vector bundle of algebraic curvature tensors over some pseudo-Riemannian manifold, see also Remark 2 below.

Proposition 1. *Let an algebraic curvature tensor R and the two-jet $(R', \nabla R', \nabla^2 R')$ of a section of the vector bundle of algebraic curvature tensors be given. Then for all $x_1, x_2, x_3, x_4 \in V$:*

$$(d^\nabla \delta^\nabla + \delta^\nabla d^\nabla) R'(x_1, x_2, x_3, x_4) = \left\{ \begin{array}{l} \nabla^* \nabla R'(x_1, x_2, x_3, x_4) + \frac{1}{2} R * R'(x_1, x_2, x_3, x_4) \\ + \frac{1}{2} \left(\begin{array}{cc} R'_{x_2, x_4} \cdot \text{ric}(x_1, x_3) & -R'_{x_2, x_3} \cdot \text{ric}(x_1, x_4) \\ +R'_{x_1, x_3} \cdot \text{ric}(x_2, x_4) & -R'_{x_1, x_4} \cdot \text{ric}(x_2, x_3) \end{array} \right) \end{array} \right\}. \quad (33)$$

Proof. It is straightforward that

$$(d^\nabla \delta^\nabla + \delta^\nabla d^\nabla) R'(x_1, x_2, x_3, x_4) = \nabla^* \nabla R'(x_1, x_2, x_3, x_4) - \sum_{i=1}^n (R_{x_1, e_i} \cdot R'(e_i, x_2, x_3, x_4) - R_{x_2, e_i} \cdot R'(e_i, x_1, x_3, x_4))$$

(cf. the proof of [3, Prop. 4.2].) Now (33) follows from (20). \square

Note, the term in (33) involving the Ricci-curvature vanishes after projection to the vector bundle of algebraic curvature tensors.

Remark 2. For every two-jet $(R', \nabla R', \nabla^2 R')$ of a section of the vector bundle of algebraic curvature tensors

$$\frac{1}{12} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} (d^\nabla \delta^\nabla + \delta^\nabla d^\nabla) R'(x_1, x_2, x_3, x_4) = \nabla^* \nabla R'(x_1, x_2, x_3, x_4) + \frac{1}{2} R * R'(x_1, x_2, x_3, x_4). \quad (34)$$

Here the Laplacian $(d^\nabla \delta^\nabla + \delta^\nabla d^\nabla)$ followed by the Young projector is a parallel second order differential operator on the vector bundle of algebraic curvature tensors $\mathcal{C}_0(TM)$ for every pseudo-Riemannian manifold M , see [10, Sec. 3]. Further, recall the decomposition of a curvature tensor

$$R = \frac{s}{2n(n-1)} \langle \cdot, \cdot \rangle \otimes \langle \cdot, \cdot \rangle + \frac{1}{n-2} g \otimes \text{ric} + W \quad (35)$$

(where \otimes denotes the Kulkarni-Nomizu product, s the scalar curvature, ric the Ricci tensor and W is the conformal Weyl tensor.) It follows from (22) and (23) that any term occurring in (34) respects the corresponding splitting of $\mathcal{C}_0(TM)$ into subbundles. Thus (34) is a Weitzenböck formula on any of these vector bundles in the strict sense of [10, Sec. 3].

Specifically, we obtain the following formula for the curvature tensor of some pseudo-Riemannian manifold:

Corollary 3. *Let a two-jet $(R, \nabla R, \nabla^2 R)$ be given. Then*

$$\nabla^* \nabla R(x_1, x_2, x_3, x_4) = \frac{1}{4} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \nabla_{x_1, x_3}^{(2)} \text{ric}(x_2, x_4) - \frac{1}{2} R * R(x_1, x_2, x_3, x_4) . \quad (36)$$

In particular, if $\nabla^2 \text{ric} = 0$ (e.g. the two-jet is Einstein), then

$$\nabla^* \nabla R(x_1, x_2, x_3, x_4) = -\frac{1}{2} R * R(x_1, x_2, x_3, x_4) . \quad (37)$$

Proof.

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \nabla_{x_1, x_3}^{(2)} \text{ric}(x_2, x_4) = \begin{cases} +2 \nabla_{x_1, x_3}^{(2)} \text{ric}(x_2, x_4) & +2 \nabla_{x_3, x_1}^{(2)} \cdot \text{ric}(x_2, x_4) \\ -2 \nabla_{x_2, x_3}^{(2)} \text{ric}(x_1, x_4) & -2 \nabla_{x_3, x_2}^{(2)} \cdot \text{ric}(x_1, x_4) \\ -2 \nabla_{x_1, x_4}^{(2)} \text{ric}(x_2, x_3) & -2 \nabla_{x_4, x_1}^{(2)} \cdot \text{ric}(x_2, x_3) \\ +2 \nabla_{x_2, x_4}^{(2)} \text{ric}(x_1, x_3) & +2 \nabla_{x_4, x_2}^{(2)} \cdot \text{ric}(x_1, x_3) \end{cases}$$

We use the Ricci identity and the second Bianchi identity via (32) to see that the last expression is

$$\begin{aligned} & \left. \begin{aligned} & +4 \nabla_{x_3, x_1}^{(2)} \text{ric}(x_2, x_4) & +2 R_{x_1, x_3} \cdot \text{ric}(x_2, x_4) \\ & -4 \nabla_{x_3, x_2}^{(2)} \text{ric}(x_1, x_4) & -2 R_{x_2, x_3} \cdot \text{ric}(x_1, x_4) \\ & -4 \nabla_{x_4, x_1}^{(2)} \text{ric}(x_2, x_3) & -2 R_{x_1, x_4} \cdot \text{ric}(x_2, x_3) \\ & +4 \nabla_{x_4, x_2}^{(2)} \text{ric}(x_1, x_3) & +2 R_{x_2, x_4} \cdot \text{ric}(x_1, x_3) \end{aligned} \right\} \stackrel{(32)}{=} \begin{cases} +4 d^\nabla \delta^\nabla R(x_3, x_4, x_1, x_2) & +4 \delta^\nabla \underbrace{d^\nabla R}_{\substack{2. \text{ Bianchi}_0 \\ \equiv}}(x_3, x_4, x_1, x_2) \\ +2 R_{x_2, x_4} \cdot \text{ric}(x_1, x_3) & -2 R_{x_1, x_4} \cdot \text{ric}(x_2, x_3) \\ +2 R_{x_1, x_3} \cdot \text{ric}(x_2, x_4) & -2 R_{x_2, x_3} \cdot \text{ric}(x_1, x_4) \end{cases} \\ & \stackrel{(33)}{=} 4 \nabla^* \nabla R(x_1, x_2, x_3, x_4) + 2 R * R(x_1, x_2, x_3, x_4) . \end{aligned}$$

The result follows. \square

4 Traces of the associated linear two-jet

Let V be a Euclidean vector space and $\{e_1, \dots, e_n\}$ be an orthonormal basis of V . Let $A \in \bigotimes^k V^*$ be a covariant k -tensor. We put

$$\text{tr}_{i,j} A(x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_k) := \sum_{i,j=1}^n A(x_1, \dots, e_i, \dots, e_j, \dots, x_k) , \quad (38)$$

the trace of A with respect to the variables x_i and x_j .

Given $\nabla^2 R \in \mathcal{C}_2$ there exist exactly three different traces: the second covariant derivative of the Ricci tensor

$$\nabla_{x_1, x_2}^{(2)} \text{ric}(y_1, y_2) = - \sum_{i=1}^n \nabla_{x_1, x_2}^{(2)} R(y_1, e_i, y_2, e_i), \quad (39)$$

the covariant derivative of the divergence

$$\nabla_x \delta_{y_1}^\nabla R(y_2, y_3) := - \sum_{i=1}^n \nabla_{x, e_i}^{(2)} R(e_i, y_1, y_2, y_3), \quad (40)$$

and the rough Laplacian given by

$$\nabla^* \nabla R(x_1, x_2, x_3, x_4) := - \sum_{i=1}^n \nabla_{e_i, e_i}^{(2)} R(x_1, x_2, x_3, x_4). \quad (41)$$

Moreover, there are the following relations between the three traces:

Lemma 4. *Let $\nabla^2 R \in \mathcal{C}_2$ be given. Then*

$$\nabla_{x_1} \delta_{x_2}^\nabla R(x_3, x_4) = \nabla_{x_1, x_3}^{(2)} \text{ric}(x_2, x_4) - \nabla_{x_1, x_4}^{(2)} \text{ric}(x_2, x_3), \quad (42)$$

$$\nabla^* \nabla R(x_1, x_2, x_3, x_4) = \frac{1}{4} \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \nabla_{x_1, x_3}^{(2)} \text{ric}(x_2, x_4) \quad (43)$$

$$= \nabla_{x_1} \delta_{x_2}^\nabla R(x_3, x_4) - \nabla_{x_2} \delta_{x_1}^\nabla R(x_3, x_4). \quad (44)$$

In particular,

$$\nabla^2 \text{ric} = 0 \xrightarrow{(42)} \nabla \delta^\nabla R = 0 \xrightarrow{(43)} \nabla^* \nabla R = 0. \quad (45)$$

Proof. The first equation uses the second Bianchi identity. Further note that (43) is a special case of (36) since $(0, 0, \nabla^2 R)$ is a two-jet by definition of \mathcal{C}_2 . Then (44) follows from (42) since $\nabla^2 \text{ric} \in \text{Sym}^2 V^* \otimes \text{Sym}^2 V^*$. \square

Thus $\nabla^2 \text{ric}$ should be seen as the *essential trace* of $\nabla^2 R$ because its vanishing already implies that $\nabla^2 R$ is totally trace-free according to (42)-(45).

4.1 The associated second covariant derivative of the Ricci tensor

For every algebraic two-jet $(R, \nabla R, \nabla^2 R)$ we consider the symmetrized/anti-symmetrized second covariant derivative of the curvature tensor

$$\tilde{\nabla}_{x_5, x_6}^{(2)} \tilde{R}(x_1, x_2, x_3, x_4) := \begin{array}{|c|c|c|c|} \hline 1 & 3 & 5 & 6 \\ \hline 2 & 4 & & \\ \hline \end{array} \nabla_{x_5, x_6}^{(2)} R(x_1, x_2, x_3, x_4). \quad (46)$$

Then $\tilde{\nabla}^{(2)} \tilde{\mathcal{C}}_2$. The associated second covariant derivative of the Ricci tensor is thus given by

$$\tilde{\nabla}_{x_5, x_6}^{(2)} \tilde{\text{ric}}(x_2, x_4) := -\text{tr}_{1,3} \begin{array}{|c|c|c|c|} \hline 1 & 3 & 5 & 6 \\ \hline 2 & 4 & & \\ \hline \end{array} \nabla_{x_5, x_6}^{(2)} R(x_1, x_2, x_3, x_4). \quad (47)$$

In view of (13) and the remarks after that, the Ricci identity implies that there exists some expression $f(R)$ quadratic in R such that

$$\tilde{\nabla}_{x_5, x_6}^2 \text{ric}(x_2, x_4) - 80 \nabla_{x_5, x_6}^2 \text{ric}(x_2, x_4) = f(R). \quad (48)$$

In fact, the explicit expression for $f(R)$ will be given in (58) below.

Lemma 5. *For every algebraic curvature tensor R on V and all $x_1, \dots, x_6 \in V$*

$$\text{tr}_{1,3} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} R_{x_5, x_3} \cdot R(x_1, x_2, x_4, x_6) = \text{tr}_{1,3} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} R_{x_5, x_1} \cdot R(x_3, x_2, x_4, x_6) \quad (49)$$

$$= 3 \begin{pmatrix} \sum_{i=1}^n (R_{x_5, e_i} \cdot R(e_i, x_2, x_4, x_6) + R_{x_5, e_i} \cdot R(e_i, x_4, x_2, x_6)) \\ + R_{x_5, x_2} \cdot \text{ric}(x_4, x_6) \quad + R_{x_5, x_4} \cdot \text{ric}(x_2, x_6) \end{pmatrix}. \quad (50)$$

Proof. Equation 51 is clear from (9) and either side of the equation is given by

$$\sum_{i=1}^n \begin{pmatrix} +R_{x_5, e_i} \cdot R(e_i, x_2, x_4, x_6) & +R_{x_5, e_i} \cdot R(e_i, x_2, x_4, x_6) & +R_{x_5, e_i} \cdot R(e_i, x_4, x_2, x_6) & +R_{x_5, e_i} \cdot R(e_i, x_4, x_2, x_6) \\ -R_{x_5, x_2} \cdot R(e_i, e_i, x_4, x_6) & -R_{x_5, e_i} \cdot R(x_2, e_i, x_4, x_6) & -R_{x_5, x_2} \cdot R(e_i, x_4, e_i, x_6) & -R_{x_5, e_i} \cdot R(x_2, x_4, e_i, x_6) \\ -R_{x_5, e_i} \cdot R(x_4, x_2, e_i, x_6) & -R_{x_5, x_4} \cdot R(e_i, x_2, e_i, x_6) & -R_{x_5, e_i} \cdot R(x_4, e_i, x_2, x_6) & -R_{x_5, x_4} \cdot R(e_i, e_i, x_2, x_6) \\ +R_{x_5, x_2} \cdot R(x_4, e_i, e_i, x_6) & +R_{x_5, x_4} \cdot R(x_2, e_i, e_i, x_6) & +R_{x_5, x_2} \cdot R(x_4, e_i, e_i, x_6) & +R_{x_5, x_4} \cdot R(x_2, e_i, e_i, x_6) \end{pmatrix}.$$

from which the result follows. \square

Corollary 4. *Let R be an algebraic curvature tensor.*

$$\text{tr}_{1,3} \begin{bmatrix} 1 & 3 & 6 \\ 2 & 4 & \end{bmatrix} R_{x_5, x_6} \cdot R(x_1, x_2, x_3, x_4) = 6 \begin{pmatrix} -2 R_{x_5, x_6} \cdot \text{ric}(x_2, x_4) \\ + \sum_{i=1}^n R_{x_5, e_i} \cdot R(e_i, x_2, x_6, x_4) \\ + \sum_{i=1}^n R_{x_5, e_i} \cdot R(e_i, x_4, x_6, x_2) \\ - R_{x_5, x_2} \cdot \text{ric}(x_4, x_6) \\ - R_{x_5, x_4} \cdot \text{ric}(x_2, x_6) \end{pmatrix}. \quad (51)$$

Proof.

$$\begin{aligned} & \text{tr}_{1,3} \begin{bmatrix} 1 & 3 & 6 \\ 2 & 4 & \end{bmatrix} R_{x_5, x_6} \cdot R(x_1, x_2, x_3, x_4) = \underbrace{\text{tr}_{1,3} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} R_{x_5, x_6} \cdot R(x_1, x_2, x_3, x_4)}_{\stackrel{(13)}{=} -12 R_{x_5, x_6} \cdot \text{ric}(x_2, x_4)} \\ & + \underbrace{\text{tr}_{1,3} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} R_{x_5, x_1} \cdot R(x_3, x_2, x_6, x_4)}_{\stackrel{(49)}{=} \begin{pmatrix} -3 \sum_{i=1}^n R_{x_5, e_i} \cdot R(e_i, x_2, x_4, x_6) \\ -3 \sum_{i=1}^n R_{x_5, e_i} \cdot R(e_i, x_4, x_2, x_6) \\ -3 R_{x_5, x_2} \cdot \text{ric}(x_4, x_6) \\ -3 R_{x_5, x_4} \cdot \text{ric}(x_2, x_6) \end{pmatrix}} \\ & + \underbrace{\text{tr}_{1,3} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} R_{x_5, x_3} \cdot R(x_6, x_2, x_1, x_4)}_{\stackrel{(49)}{=} \begin{pmatrix} -3 \sum_{i=1}^n R_{x_5, e_i} \cdot R(e_i, x_2, x_4, x_6) \\ -3 \sum_{i=1}^n R_{x_5, e_i} \cdot R(e_i, x_4, x_2, x_6) \\ -3 R_{x_5, x_2} \cdot \text{ric}(x_4, x_6) \\ -3 R_{x_5, x_4} \cdot \text{ric}(x_2, x_6) \end{pmatrix}}. \end{aligned}$$

\square

For the following lemma let S_I denote the Permutation group of some set I .

Lemma 6. Let some algebraic two-jet $(R, \nabla R, \nabla^2 R)$ be given.

$$- \operatorname{tr}_{1,3} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \nabla_{x_1, x_3}^{(2)} R(x_5, x_2, x_6, x_4) = \sum_{(\sigma, \tau) \in \mathcal{S}_{\{2,4\}} \times \mathcal{S}_{\{5,6\}}} \begin{cases} \nabla^* \nabla R(x_{\tau(5)}, x_{\sigma(2)}, x_{\tau(6)}, x_{\sigma(4)}) \\ + \nabla_{x_{\sigma(2)}, x_{\sigma(4)}}^{(2)} \operatorname{ric}(x_{\tau(5)}, x_{\tau(6)}) \\ - 2 \nabla_{x_{\sigma(2)}} \delta_{x_{\tau(5)}} R(x_{\sigma(4)}, x_{\tau(6)}) \\ - \sum_{i=1}^n R_{x_{\sigma(2)}, e_i} \cdot R(e_i, x_{\tau(5)}, x_{\sigma(4)}, x_{\tau(6)}) \end{cases} \quad (52)$$

Proof. $\operatorname{tr}_{1,3} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \nabla_{x_1, x_3}^{(2)} R(x_5, x_2, x_6, x_4)$

$$= \sum_{i=1}^n \begin{cases} + \nabla_{e_i, e_i}^{(2)} R(x_5, x_2, x_6, x_4) & + \nabla_{e_i, e_i}^{(2)} R(x_5, x_2, x_6, x_4) & + \nabla_{e_i, e_i}^{(2)} R(x_5, x_4, x_6, x_2) & + \nabla_{e_i, e_i}^{(2)} R(x_5, x_4, x_6, x_2) \\ - \nabla_{x_2, e_i}^{(2)} R(x_5, e_i, x_6, x_4) & - \nabla_{e_i, x_2}^{(2)} R(x_5, e_i, x_6, x_4) & - \nabla_{x_2, e_i}^{(2)} R(x_5, x_4, x_6, e_i) & - \nabla_{e_i, x_2}^{(2)} R(x_5, x_4, x_6, e_i) \\ - \nabla_{e_i, x_4}^{(2)} R(x_5, x_2, x_6, e_i) & - \nabla_{x_4, e_i}^{(2)} R(x_5, x_2, x_6, e_i) & - \nabla_{e_i, x_4}^{(2)} R(x_5, e_i, x_6, x_2) & - \nabla_{x_4, e_i}^{(2)} R(x_5, e_i, x_6, x_2) \\ + \nabla_{x_2, x_4}^{(2)} R(x_5, e_i, x_6, e_i) & + \nabla_{x_4, x_2}^{(2)} R(x_5, e_i, x_6, e_i) & + \nabla_{x_2, x_4}^{(2)} R(x_5, e_i, x_6, e_i) & + \nabla_{x_4, x_2}^{(2)} R(x_5, e_i, x_6, e_i) \end{cases}$$

$$= \begin{cases} -2 \nabla^* \nabla R(x_5, x_2, x_6, x_4) & -2 \nabla^* \nabla R(x_5, x_4, x_6, x_2) \\ +2 \nabla_{x_2} \delta_{x_5} R(x_4, x_6) & + \sum_{i=1}^n R_{x_2, e_i} \cdot R(e_i, x_5, x_4, x_6) \\ +2 \nabla_{x_2} \delta_{x_6} R(x_4, x_5) & + \sum_{i=1}^n R_{x_2, e_i} \cdot R(e_i, x_6, x_4, x_5) \\ +2 \nabla_{x_4} \delta_{x_6} R(x_2, x_5) & + \sum_{i=1}^n R_{x_4, e_i} \cdot R(e_i, x_6, x_2, x_5) \\ +2 \nabla_{x_4} \delta_{x_5} R(x_2, x_6) & + \sum_{i=1}^n R_{x_4, e_i} \cdot R(e_i, x_5, x_2, x_6) \\ -2 \nabla_{x_2, x_4}^{(2)} \operatorname{ric}(x_5, x_6) & -2 \nabla_{x_4, x_2}^{(2)} \operatorname{ric}(x_5, x_6) \end{cases}.$$

□

Proposition 2. For an algebraic two-jet $(R, \nabla R, \nabla^2 R)$ consider the associated second covariant derivative of the Ricci tensor (47). We have

$$\tilde{\nabla}_{x_5, x_6}^{(2)} \tilde{\operatorname{ric}}(x_2, x_4) = 2 \sum_{(\sigma, \tau) \in \mathcal{S}_{\{2,4\}} \times \mathcal{S}_{\{5,6\}}} \begin{cases} -R * R(x_{\tau(5)}, x_{\sigma(2)}, x_{\tau(6)}, x_{\sigma(4)}) \\ + 2 R_{x_{\tau(5)}, x_{\sigma(2)}} \cdot \operatorname{ric}(x_{\tau(6)}, x_{\sigma(4)}) \\ + 10 \nabla_{x_{\tau(5)}, x_{\tau(6)}}^{(2)} \operatorname{ric}(x_{\sigma(2)}, x_{\sigma(4)}) \end{cases} \quad (53)$$

Proof. Let $\mathcal{S}_{\{1,3,5,6\}}$, $\mathcal{S}_{\{1,3,5\}}$, $\mathcal{S}_{\{1,3,6\}}$ and $\mathcal{S}_{\{5,6\}}$ denote the symmetric groups over $\{1, 3, 5, 6\}$, $\{1, 3, 5\}$, $\{1, 3, 6\}$ and $\{5, 6\}$, respectively. In particular, $\mathcal{S}_{\{5,6\}}$ is a subgroup of $\mathcal{S}_{\{1,3,5,6\}}$. Let $\mathcal{S}_{\{5,6\}} \backslash \mathcal{S}_{\{1,3,5,6\}}$ be the set of right cosets. In the free vector space over $\mathcal{S}_{\{5,6\}} \backslash \mathcal{S}_{\{1,3,5,6\}}$

$$\sum_{[\pi] \in \mathcal{S}_{\{5,6\}} \backslash \mathcal{S}_{\{1,3,5,6\}}} [\pi] = \sum_{\pi \in \mathcal{S}_{\{1,3,6\}}} [\pi] + \sum_{\pi \in \mathcal{S}_{\{1,3,5\}}} [\pi] - \sum_{\pi \in \mathcal{S}_{\{1,3\}}} [\pi] + [(15)(36)] + [(16)(35)] \quad (54)$$

(see [7, Eq. 46 on p. 13].) Thus l.h.s. of (53) is given by

$$\begin{aligned}
& -\text{tr}_{1,3} \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 4 & \\ \hline \end{array} \underbrace{\nabla_{x_6,x_5}^{(2)} R(x_1, x_2, x_3, x_4) + \nabla_{x_5,x_6}^{(2)} R(x_1, x_2, x_3, x_4)}_{=2 \nabla_{x_6,x_5}^{(2)} R(x_1, x_2, x_3, x_4) + R_{x_5,x_6} \cdot R(x_1, x_2, x_3, x_4)} \\
& \stackrel{(13),(51)}{=} 48 \nabla_{x_6,x_5}^{(2)} \text{ric}(x_2, x_4) + 12 R_{x_6,x_5} \cdot \text{ric}(x_2, x_4) + 6 \left\{ \begin{array}{l} + \sum_{i=1}^n R_{x_6,e_i} \cdot R(e_i, x_2, x_5, x_4) \\ + \sum_{i=1}^n R_{x_6,e_i} \cdot R(e_i, x_4, x_5, x_2) \\ + \sum_{i=1}^n R_{x_6,x_2} \cdot \text{ric}(x_4, x_5) \\ + \sum_{i=1}^n R_{x_6,x_4} \cdot \text{ric}(x_2, x_5) \end{array} \right. \\
& -\text{tr}_{1,3} \begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 4 & \\ \hline \end{array} \underbrace{\nabla_{x_5,x_6}^{(2)} R(x_1, x_2, x_3, x_4) + \nabla_{x_6,x_5}^{(2)} R(x_1, x_2, x_3, x_4)}_{=2 \nabla_{x_5,x_6}^{(2)} R(x_1, x_2, x_3, x_4) + R_{x_6,x_5} \cdot R(x_1, x_2, x_3, x_4)} \\
& \stackrel{(13),(51)}{=} 48 \nabla_{x_5,x_6}^{(2)} \text{ric}(x_2, x_4) + 12 R_{x_5,x_6} \cdot \text{ric}(x_2, x_4) + 6 \left\{ \begin{array}{l} + \sum_{i=1}^n R_{x_5,e_i} \cdot R(e_i, x_2, x_6, x_4) \\ + \sum_{i=1}^n R_{x_5,e_i} \cdot R(e_i, x_4, x_6, x_2) \\ + R_{x_5,x_2} \cdot \text{ric}(x_4, x_6) \\ + R_{x_5,x_4} \cdot \text{ric}(x_2, x_6) \end{array} \right. \\
& + \text{tr}_{1,3} \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \underbrace{\nabla_{x_5,x_6}^{(2)} R(x_1, x_2, x_3, x_4) + \nabla_{x_6,x_5}^{(2)} R(x_1, x_2, x_3, x_4)}_{\stackrel{(13)}{=} -12(\nabla_{x_5,x_6}^{(2)} \text{ric}(x_2, x_4) + \nabla_{x_6,x_5}^{(2)} \text{ric}(x_2, x_4))} \\
& -\text{tr}_{1,3} \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \underbrace{(\nabla_{x_1,x_3}^{(2)} R(x_5, x_2, x_6, x_4) + \nabla_{x_1,x_3}^{(2)} R(x_6, x_2, x_5, x_4))}_{\stackrel{(52)}{=} 2 \left(\begin{array}{ll} +2 \nabla^* \nabla R(x_5, x_2, x_6, x_4) & +2 \nabla^* \nabla R(x_6, x_2, x_5, x_4) \\ -2 \nabla_{x_2} \delta_{x_5} R(x_4, x_6) & - \sum_{i=1}^n R_{x_2,e_i} \cdot R(e_i, x_5, x_4, x_6) \\ -2 \nabla_{x_2} \delta_{x_6} R(x_4, x_5) & - \sum_{i=1}^n R_{x_2,e_i} \cdot R(e_i, x_6, x_4, x_5) \\ -2 \nabla_{x_4} \delta_{x_6} R(x_2, x_5) & - \sum_{i=1}^n R_{x_4,e_i} \cdot R(e_i, x_6, x_2, x_5) \\ -2 \nabla_{x_4} \delta_{x_5} R(x_2, x_6) & - \sum_{i=1}^n R_{x_4,e_i} \cdot R(e_i, x_5, x_2, x_6) \\ +2 \nabla_{x_2,x_4}^{(2)} \text{ric}(x_5, x_6) & +2 \nabla_{x_4,x_2}^{(2)} \text{ric}(x_5, x_6) \end{array} \right)} .
\end{aligned}$$

Further, recall that

$$\begin{aligned}
& \frac{1}{4} \sum_{(\sigma, \tau) \in \mathcal{S}_{\{2,4\}} \times \mathcal{S}_{\{5,6\}}} R * R(x_{\tau(6)}, x_{\sigma(2)}, x_{\tau(5)}, x_{\sigma(4)}) \quad (55)
\end{aligned}$$

$$\stackrel{(24)}{=} - \sum_{i=1}^n \left\{ \begin{array}{ll} R_{x_6,e_i} \cdot R(e_i, x_2, x_5, x_4) & + R_{x_6,e_i} \cdot R(e_i, x_4, x_5, x_2) \\ + R_{x_5,e_i} \cdot R(e_i, x_2, x_6, x_4) & + R_{x_5,e_i} \cdot R(e_i, x_4, x_6, x_2) \end{array} \right. \quad (56)$$

$$\stackrel{(24)}{=} - \sum_{i=1}^n \left\{ \begin{array}{ll} R_{x_2,e_i} \cdot R(e_i, x_5, x_4, x_6) & + R_{x_2,e_i} \cdot R(e_i, x_6, x_4, x_5) \\ + R_{x_4,e_i} \cdot R(e_i, x_6, x_2, x_5) & + R_{x_4,e_i} \cdot R(e_i, x_5, x_2, x_6) \end{array} \right. . \quad (57)$$

We conclude that

$$\tilde{\nabla}_{x_5, x_6}^{(2)} \tilde{\text{ric}}(x_2, x_4) = \sum_{(\sigma, \tau) \in \mathcal{S}_{\{2,4\}} \times \mathcal{S}_{\{5,6\}}} \left\{ \begin{array}{l} 2 \nabla^* \nabla R(x_{\tau(5)}, x_{\sigma(2)}, x_{\tau(6)}, x_{\sigma(4)}) \\ - R * R(x_{\tau(5)}, x_{\sigma(2)}, x_{\tau(6)}, x_{\sigma(4)}) \\ + 6 R_{x_{\tau(5)}, x_{\sigma(2)}} \cdot \text{ric}(x_{\tau(6)}, x_{\sigma(4)}) \\ - 4 \nabla_{x_{\sigma(2)}} \delta_{x_{\tau(5)}}^{\nabla} R(x_{\sigma(4)}, x_{\tau(6)}) \\ + 2 \nabla_{x_{\sigma(2)}, x_{\sigma(4)}}^{(2)} \text{ric}(x_{\tau(5)}, x_{\tau(6)}) \\ + 18 \nabla_{x_{\tau(5)}, x_{\tau(6)}}^{(2)} \text{ric}(x_{\sigma(2)}, x_{\sigma(4)}) \end{array} \right. .$$

Further,

$$\begin{aligned} & \sum_{(\sigma, \tau) \in \mathcal{S}_{\{2,4\}} \times \mathcal{S}_{\{5,6\}}} \nabla^* \nabla R(x_{\tau(5)}, x_{\sigma(2)}, x_{\tau(6)}, x_{\sigma(4)}) \\ & \stackrel{(33)}{=} \sum_{(\sigma, \tau) \in \mathcal{S}_{\{2,4\}} \times \mathcal{S}_{\{5,6\}}} \left(\underbrace{\text{d}^{\nabla} \delta_{x_{\tau(5)}, x_{\sigma(2)}}^{\nabla} R(x_{\tau(6)}, x_{\sigma(4)})}_{\begin{array}{l} - \nabla_{x_{\sigma(2)}} \delta_{x_{\tau(5)}}^{\nabla} R(x_{\tau(6)}, x_{\sigma(4)}) \\ + \nabla_{x_{\tau(5)}, x_{\tau(6)}}^2 \text{ric}(x_{\sigma(2)}, x_{\sigma(4)}) \\ - \nabla_{x_{\tau(5)}, x_{\sigma(4)}}^2 \text{ric}(x_{\sigma(2)}, x_{\tau(6)}) \end{array}} - \frac{1}{2} R * R(x_{\tau(5)}, x_{\sigma(2)}, x_{\tau(6)}, x_{\sigma(4)}) \right) . \end{aligned}$$

We thus see that

$$\begin{aligned} \tilde{\nabla}_{x_5, x_6}^{(2)} \tilde{\text{ric}}(x_2, x_4) &= 2 \sum_{(\sigma, \tau) \in \mathcal{S}_{\{2,4\}} \times \mathcal{S}_{\{5,6\}}} \left\{ \begin{array}{l} \underbrace{- R * R(x_{\tau(5)}, x_{\sigma(2)}, x_{\tau(6)}, x_{\sigma(4)}) + 3 R_{x_{\tau(5)}, x_{\sigma(2)}} \cdot \text{ric}(x_{\tau(6)}, x_{\sigma(4)})}_{= \nabla_{x_{\sigma(2)}, x_{\tau(6)}}^{(2)} \text{ric}(x_{\tau(5)}, x_{\sigma(4)})} \\ - \nabla_{x_{\sigma(2)}} \delta_{x_{\tau(5)}}^{\nabla} R(x_{\sigma(4)}, x_{\tau(6)}) + \nabla_{x_{\sigma(2)}, x_{\sigma(4)}}^{(2)} \text{ric}(x_{\tau(5)}, x_{\tau(6)}) \\ - \nabla_{x_{\tau(5)}, x_{\sigma(4)}}^2 \text{ric}(x_{\sigma(2)}, x_{\tau(6)}) \\ + 10 \nabla_{x_{\tau(5)}, x_{\tau(6)}}^{(2)} \text{ric}(x_{\sigma(2)}, x_{\sigma(4)}) \end{array} \right\} \\ &= 2 \sum_{(\sigma, \tau) \in \mathcal{S}_{\{2,4\}} \times \mathcal{S}_{\{5,6\}}} \left\{ \begin{array}{l} \underbrace{- R * R(x_{\tau(5)}, x_{\sigma(2)}, x_{\tau(6)}, x_{\sigma(4)}) + 3 R_{x_{\tau(5)}, x_{\sigma(2)}} \cdot \text{ric}(x_{\tau(6)}, x_{\sigma(4)})}_{= \nabla_{x_{\sigma(2)}, x_{\tau(6)}}^{(2)} \text{ric}(x_{\tau(5)}, x_{\sigma(4)})} \\ + \nabla_{x_{\sigma(2)}, x_{\tau(6)}}^{(2)} \text{ric}(x_{\tau(5)}, x_{\sigma(4)}) - \nabla_{x_{\tau(5)}, x_{\sigma(4)}}^2 \text{ric}(x_{\sigma(2)}, x_{\tau(6)}) \\ + 10 \nabla_{x_{\tau(5)}, x_{\tau(6)}}^{(2)} \text{ric}(x_{\sigma(2)}, x_{\sigma(4)}) \end{array} \right\} \end{aligned}$$

The result follows from Lemma 3 together with (47). \square

We thus conclude from (53) in combination with the Ricci identity that (48) is solved by

$$f(R) := -80 R_{x_5, x_6} \cdot \text{ric}(x_2, x_4) + 2 \sum_{(\sigma, \tau) \in \mathcal{S}_{\{2,4\}} \times \mathcal{S}_{\{5,6\}}} \left\{ \begin{array}{l} - R * R(x_{\tau(5)}, x_{\sigma(2)}, x_{\tau(6)}, x_{\sigma(4)}) \\ + 2 R_{x_{\tau(5)}, x_{\sigma(2)}} \cdot \text{ric}(x_{\tau(6)}, x_{\sigma(4)}) \end{array} \right. . \quad (58)$$

Using the special case (37) of the Weitzenböck formula we conclude from Proposition 2:

Corollary 5. *Let an algebraic Einstein two-jet $(R, \nabla R, \nabla^2 R)$ be given. Then*

$$\tilde{\nabla}_{x_5, x_6}^{(2)} \tilde{\text{ric}}(x_2, x_4) = -4 (R * R(x_5, x_2, x_6, x_4) + R * R(x_5, x_4, x_6, x_2)) . \quad (59)$$

4.2 The associated rough Laplacian

Let an algebraic two-jet $(R, \nabla R, \nabla^2)$ be given. We aim to calculate the rough Laplacian associated with (46) according to (41), i.e. the algebraic curvature tensor given by

$$\tilde{\nabla}^* \tilde{\nabla} \tilde{R}(x_1, x_2, x_3, x_4) := -\text{tr}_{5,6} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \nabla_{x_5, x_6}^{(2)} R(x_1, x_2, x_3, x_4). \quad (60)$$

Clearly, we will use (44) to achieve this goal. Again it is a priori clear that there exists some expression $g(R)$ quadratic in R such that

$$\tilde{\nabla}^* \tilde{\nabla}_{x_5, x_6} \text{ric}(x_2, x_4) - 80 \nabla^* \nabla_{x_5, x_6} \text{ric}(x_2, x_4) = g(R). \quad (61)$$

Lemma 7.

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} R_{x_1, x_2} \cdot \text{ric}(x_3, x_4) = 0. \quad (62)$$

Proof.

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} R_{x_2, x_1} \cdot \text{ric}(x_3, x_4) = \left\{ \begin{array}{l} R_{x_1, x_2} \cdot \text{ric}(x_3, x_4) + R_{x_1, x_4} \cdot \text{ric}(x_3, x_2) \\ + R_{x_3, x_2} \cdot \text{ric}(x_1, x_4) + R_{x_3, x_4} \cdot \text{ric}(x_1, x_2) \\ - R_{x_2, x_1} \cdot \text{ric}(x_3, x_4) - R_{x_2, x_4} \cdot \text{ric}(x_3, x_1) \\ - R_{x_3, x_1} \cdot \text{ric}(x_2, x_4) - R_{x_3, x_4} \cdot \text{ric}(x_2, x_1) \\ - R_{x_1, x_2} \cdot \text{ric}(x_4, x_3) - R_{x_1, x_3} \cdot \text{ric}(x_4, x_2) \\ - R_{x_4, x_2} \cdot \text{ric}(x_1, x_3) - R_{x_4, x_3} \cdot \text{ric}(x_1, x_2) \\ + R_{x_2, x_1} \cdot \text{ric}(x_4, x_3) + R_{x_2, x_3} \cdot \text{ric}(x_4, x_1) \\ + R_{x_4, x_1} \cdot \text{ric}(x_2, x_3) + R_{x_4, x_3} \cdot \text{ric}(x_2, x_1) \end{array} \right\} = 0.$$

□

Proposition 3. Let an algebraic two-jet $(R, \nabla R, \nabla^2 R)$ be given. Consider the rough Laplacian (41) associated with the linear two-jet (46). We have

$$\tilde{\nabla}^* \tilde{\nabla} \tilde{R}(x_1, x_2, x_3, x_4) = 80 \nabla^* \nabla R(x_1, x_2, x_3, x_4) + 16 R * R(x_1, x_2, x_3, x_4). \quad (63)$$

Proof. According to the last theorem, (36) and Lemma 7 left hand side of (63) is given by

$$\begin{aligned} & \tilde{\nabla}^* \tilde{\nabla} \tilde{R}(x_1, x_2, x_3, x_4) \\ & \stackrel{(53)}{=} -2 \underbrace{\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} R * R(x_1, x_2, x_3, x_4)}_{\stackrel{(13)}{=} -24 R * R(x_1, x_2, x_3, x_4)} \\ & + \underbrace{20 \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \nabla_{x_1, x_3}^{(2)} \text{ric}(x_2, x_4)}_{\stackrel{(36)}{=} 80 \nabla^* \nabla R(x_1, x_2, x_3, x_4) + 40 R * R(x_1, x_2, x_3, x_4)} \\ & \stackrel{(43)}{=} \frac{1}{4} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \tilde{\nabla}_{x_1, x_3}^{(2)} \tilde{\text{ric}}(x_2, x_4) \\ & + 4 \underbrace{\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} R_{x_1, x_2} \cdot \text{ric}(x_3, x_4)}_{\stackrel{(62)}{=} 0} \end{aligned}$$

The result follows. □

5 The canonical embedding of Ricci-flat curvature tensors

Given the algebraic curvature tensor R , we set

$$\hat{\nabla}_{x_5, x_6}^{(2)} \hat{R}(x_1, x_2, x_3, x_4) := \begin{array}{|c|c|c|c|} \hline 1 & 3 & 5 & 6 \\ \hline 2 & 4 & & \\ \hline \end{array} \langle x_5, x_6 \rangle R(x_1, x_2, x_3, x_4) . \quad (64)$$

Then $\hat{\nabla}^{(2)} \hat{R} \in \mathcal{C}_2$ and the linear map

$$\iota : \mathcal{C}_0 \rightarrow \mathcal{C}_2, R \mapsto \hat{\nabla}^{(2)} \hat{R} \quad (65)$$

is an embedding according to the arguments given in the proof of Lemma 2. In the following we suppose further that $\text{ric} = 0$. By definition, the second covariant derivative of the Ricci-tensor and the rough Laplacian associated with $\hat{\nabla}^{(2)} \hat{R}$ according to (39) are given by

$$\hat{\nabla}_{x_5, x_6}^{(2)} \hat{\text{ric}}(x_2, x_4) = -\text{tr}_{1,3} \begin{array}{|c|c|c|c|} \hline 1 & 3 & 5 & 6 \\ \hline 2 & 4 & & \\ \hline \end{array} \langle x_5, x_6 \rangle R(x_1, x_2, x_3, x_4) , \quad (66)$$

$$\hat{\nabla}^* \hat{\nabla} \hat{R}(x_1, x_2, x_3, x_4) = -\text{tr}_{5,6} \begin{array}{|c|c|c|c|} \hline 1 & 3 & 5 & 6 \\ \hline 2 & 4 & & \\ \hline \end{array} \langle x_5, x_6 \rangle R(x_1, x_2, x_3, x_4) . \quad (67)$$

Lemma 8. *Let an algebraic curvature tensor R with vanishing Ricci tensor be given. Then*

$$\text{tr}_{1,3} \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \langle x_1, x_6 \rangle R(x_3, x_2, x_5, x_4) = 3 (R(x_6, x_2, x_5, x_4) + R(x_6, x_4, x_5, x_2)) . \quad (68)$$

Proof.

$$\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \langle x_1, x_6 \rangle R(x_3, x_2, x_5, x_4) = \left\{ \begin{array}{ll} +\langle x_1, x_6 \rangle R(x_3, x_2, x_5, x_4) & +\langle x_3, x_6 \rangle R(x_1, x_2, x_5, x_4) \\ +\langle x_1, x_6 \rangle R(x_3, x_4, x_5, x_2) & +\langle x_3, x_6 \rangle R(x_1, x_4, x_5, x_2) \\ -\langle x_2, x_6 \rangle R(x_3, x_1, x_5, x_4) & -\langle x_3, x_6 \rangle R(x_2, x_1, x_5, x_4) \\ -\langle x_2, x_6 \rangle R(x_3, x_4, x_5, x_1) & -\langle x_3, x_6 \rangle R(x_2, x_4, x_5, x_1) \\ -\langle x_1, x_6 \rangle R(x_4, x_2, x_5, x_3) & -\langle x_4, x_6 \rangle R(x_1, x_2, x_5, x_3) \\ -\langle x_1, x_6 \rangle R(x_4, x_3, x_5, x_2) & -\langle x_4, x_6 \rangle R(x_1, x_3, x_5, x_2) \\ +\langle x_2, x_6 \rangle R(x_4, x_1, x_5, x_3) & +\langle x_4, x_6 \rangle R(x_2, x_1, x_5, x_3) \\ +\langle x_2, x_6 \rangle R(x_4, x_3, x_5, x_1) & +\langle x_4, x_6 \rangle R(x_2, x_3, x_5, x_1) \end{array} \right. .$$

Hence,

$$\begin{aligned} \text{tr}_{1,3} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \langle x_1, x_6 \rangle R(x_3, x_2, x_4, x_5) &= \begin{cases} +\langle e_i, x_6 \rangle R(e_i, x_2, x_5, x_4) & +\langle e_i, x_6 \rangle R(e_i, x_2, x_5, x_4) \\ +\langle e_i, x_6 \rangle R(e_i, x_4, x_5, x_2) & +\langle e_i, x_6 \rangle R(e_i, x_4, x_5, x_2) \\ -\langle x_2, x_6 \rangle R(e_i, e_i, x_5, x_4) & -\langle e_i, x_6 \rangle R(x_2, e_i, x_5, x_4) \\ -\langle x_2, x_6 \rangle R(e_i, x_4, x_5, e_i) & -\langle e_i, x_6 \rangle R(x_2, x_4, x_5, e_i) \\ -\langle e_i, x_6 \rangle R(x_4, x_2, x_5, e_i) & -\langle x_4, x_6 \rangle R(e_i, x_2, x_5, e_i) \\ -\langle e_i, x_6 \rangle R(x_4, e_i, x_5, x_2) & -\langle x_4, x_6 \rangle R(e_i, e_i, x_5, x_2) \\ +\langle x_2, x_6 \rangle R(x_4, e_i, x_5, e_i) & +\langle x_4, x_6 \rangle R(x_2, e_i, x_5, e_i) \\ +\langle x_2, x_6 \rangle R(x_4, e_i, x_5, e_i) & +\langle x_4, x_6 \rangle R(x_2, e_i, x_5, e_i) \end{cases} \\ &= \begin{cases} +R(x_6, x_2, x_5, x_4) & +R(x_6, x_2, x_5, x_4) \\ +R(x_6, x_4, x_5, x_2) & +R(x_6, x_4, x_5, x_2) \\ & -R(x_2, x_6, x_5, x_4) \\ -\langle x_2, x_6 \rangle \text{ric}(x_4, x_5) & -R(x_2, x_4, x_5, x_6) \\ -R(x_4, x_2, x_5, x_6) & -\langle x_4, x_6 \rangle \text{ric}(x_2, x_5) \\ -R(x_4, x_6, x_5, x_2) & \\ -\langle x_2, x_6 \rangle \text{ric}(x_4, x_5) & -\langle x_4, x_6 \rangle \text{ric}(x_2, x_5) \\ -\langle x_2, x_6 \rangle \text{ric}(x_4, x_5) & -\langle x_4, x_6 \rangle \text{ric}(x_2, x_5) \end{cases} = 3 \begin{pmatrix} +R(x_6, x_2, x_5, x_4) & +R(x_6, x_4, x_5, x_2) \\ -\langle x_2, x_6 \rangle \text{ric}(x_4, x_5) & -\langle x_4, x_6 \rangle \text{ric}(x_2, x_5) \end{pmatrix}. \end{aligned}$$

□

Corollary 6. *Let an algebraic curvature tensor R with vanishing Ricci tensor be given. Then*

$$\text{tr}_{1,3} \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & \end{bmatrix} \langle x_5, x_6 \rangle R(x_1, x_2, x_3, x_4) = 6 \left(R(x_6, x_2, x_5, x_4) + R(x_6, x_4, x_5, x_2) \right). \quad (69)$$

Proof.

$$\begin{aligned} \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & \end{bmatrix} \langle x_5, x_6 \rangle R(x_1, x_2, x_3, x_4) &= \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \langle x_5, x_6 \rangle R(x_1, x_2, x_3, x_4) \\ + \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \langle x_1, x_6 \rangle R(x_3, x_2, x_5, x_4) &+ \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \langle x_3, x_6 \rangle R(x_5, x_2, x_1, x_4). \end{aligned}$$

Hence,

$$\begin{aligned} \text{tr}_{1,3} \begin{bmatrix} 1 & 3 & 3 \\ 2 & 4 & \end{bmatrix} \langle x_5, x_6 \rangle R(x_1, x_2, x_3, x_4) &= \underbrace{\text{tr}_{1,3} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \langle x_5, x_6 \rangle R(x_1, x_2, x_3, x_4)}_{\stackrel{(13)}{=} 0} \\ + \underbrace{\text{tr}_{1,3} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \langle x_1, x_6 \rangle R(x_3, x_2, x_5, x_4)}_{\stackrel{(68)}{=} +3R(x_6, x_2, x_5, x_4) + 3R(x_6, x_4, x_5, x_2)} &+ \underbrace{\text{tr}_{1,3} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \langle x_3, x_6 \rangle R(x_5, x_2, x_1, x_4)}_{\stackrel{(68)}{=} +3R(x_6, x_2, x_5, x_4) + 3R(x_6, x_4, x_5, x_2)}. \end{aligned}$$

The result follows. □

Lemma 9. *Let an algebraic curvature tensor R with $\text{ric} = 0$ be given. Then*

$$\text{tr}_{1,3} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \langle x_1, x_3 \rangle R(x_5, x_2, x_6, x_4) = (2n - 4) \left(R(x_5, x_2, x_6, x_4) + R(x_5, x_4, x_6, x_2) \right). \quad (70)$$

Proof.

$$\begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \langle x_1, x_3 \rangle R(x_5, x_2, x_6, x_4) = \begin{cases} +\langle x_1, x_3 \rangle R(x_5, x_2, x_6, x_4) & +\langle x_3, x_1 \rangle R(x_5, x_2, x_6, x_4) \\ +\langle x_1, x_3 \rangle R(x_5, x_4, x_6, x_2) & +\langle x_3, x_1 \rangle R(x_5, x_4, x_6, x_2) \\ -\langle x_2, x_3 \rangle R(x_5, x_1, x_6, x_4) & -\langle x_3, x_2 \rangle R(x_5, x_1, x_6, x_4) \\ -\langle x_2, x_3 \rangle R(x_5, x_4, x_6, x_1) & -\langle x_3, x_2 \rangle R(x_5, x_4, x_6, x_1) \\ -\langle x_1, x_4 \rangle R(x_5, x_2, x_6, x_3) & -\langle x_4, x_1 \rangle R(x_5, x_2, x_6, x_3) \\ -\langle x_1, x_4 \rangle R(x_5, x_3, x_6, x_2) & -\langle x_4, x_1 \rangle R(x_5, x_3, x_6, x_2) \\ +\langle x_2, x_4 \rangle R(x_5, x_1, x_6, x_3) & +\langle x_4, x_2 \rangle R(x_5, x_1, x_6, x_3) \\ +\langle x_2, x_4 \rangle R(x_5, x_3, x_6, x_1) & +\langle x_4, x_2 \rangle R(x_5, x_3, x_6, x_1) \end{cases}.$$

Hence,

$$\begin{aligned} \text{tr}_{1,3} \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \langle x_1, x_3 \rangle R(x_5, x_2, x_4, x_6) &= \begin{cases} +\langle e_i, e_i \rangle R(x_5, x_2, x_6, x_4) & +\langle e_i, e_i \rangle R(x_5, x_2, x_6, x_4) \\ +\langle e_i, e_i \rangle R(x_5, x_4, x_6, x_2) & +\langle e_i, e_i \rangle R(x_5, x_4, x_6, x_2) \\ -\langle x_2, e_i \rangle R(x_5, e_i, x_6, x_4) & -\langle e_i, x_2 \rangle R(x_5, e_i, x_6, x_4) \\ -\langle x_2, e_i \rangle R(x_5, x_4, x_6, e_i) & -\langle e_i, x_2 \rangle R(x_5, x_4, x_6, e_i) \\ -\langle e_i, x_4 \rangle R(x_5, x_2, x_6, e_i) & -\langle x_4, e_i \rangle R(x_5, x_2, x_6, e_i) \\ -\langle e_i, x_4 \rangle R(x_5, e_i, x_6, x_2) & -\langle x_4, e_i \rangle R(x_5, e_i, x_6, x_2) \\ +\langle x_2, x_4 \rangle R(x_5, e_i, x_6, e_i) & +\langle x_4, x_2 \rangle R(x_5, e_i, x_6, e_i) \\ +\langle x_2, x_4 \rangle R(x_5, e_i, x_6, e_i) & +\langle x_4, x_2 \rangle R(x_5, e_i, x_6, e_i) \end{cases} \\ &= 2 \begin{pmatrix} +nR(x_5, x_2, x_6, x_4) & +nR(x_5, x_4, x_6, x_2) \\ -R(x_5, x_2, x_6, x_4) & -R(x_5, x_4, x_6, x_2) \\ -R(x_5, x_2, x_6, x_4) & -R(x_5, x_4, x_6, x_2) \\ -\langle x_2, x_4 \rangle \text{ric}(x_5, x_6) & -\langle x_2, x_4 \rangle \text{ric}(x_5, x_6) \end{pmatrix} = \begin{cases} +(2n-4)R(x_5, x_2, x_4, x_6) \\ +(2n-4)R(x_5, x_4, x_2, x_6) \\ +4\langle x_2, x_4 \rangle \text{ric}(x_5, x_6) \end{cases}. \end{aligned}$$

□

Corollary 7. *Let an algebraic curvature tensor R with vanishing Ricci tensor be given. Then the second covariant derivative of the Ricci-tensor and the rough Laplacian associated with (64) are given by*

$$(a) \quad \hat{\nabla}_{x_5, x_6}^{(2)} \hat{\text{ric}}(x_2, x_4) = -4(n+4)(R(x_5, x_2, x_6, x_4) + R(x_5, x_4, x_6, x_2)), \quad (71)$$

$$(b) \quad \hat{\nabla}^* \hat{\nabla} \hat{R}(x_1, x_2, x_3, x_4) = -24(n+4) R(x_1, x_2, x_3, x_4). \quad (72)$$

Proof. Using the previous together with (54),

$$\begin{aligned} \frac{1}{2} \text{tr}_{1,3} \begin{array}{|c|c|c|c|} \hline 1 & 3 & 5 & 6 \\ \hline 2 & 4 & & \\ \hline \end{array} \langle x_5, x_6 \rangle R(x_1, x_2, x_3, x_4) &= \underbrace{\text{tr}_{1,3} \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 4 & \\ \hline \end{array} \langle x_5, x_6 \rangle R(x_1, x_2, x_3, x_4)}_{\stackrel{(69)}{=} 6R(x_6, x_2, x_5, x_4) + 6R(x_6, x_4, x_5, x_2)} \\ &+ \underbrace{\text{tr}_{1,3} \begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 4 & \\ \hline \end{array} \langle x_5, x_6 \rangle R(x_1, x_2, x_3, x_4)}_{\stackrel{(69)}{=} 6R(x_5, x_2, x_6, x_4) + 6R(x_5, x_4, x_6, x_2)} \\ &+ \underbrace{\text{tr}_{1,3} \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \langle x_1, x_3 \rangle R(x_5, x_2, x_6, x_4)}_{\stackrel{(70)}{=} (2n-4)R(x_5, x_2, x_6, x_4) + (2n-4)R(x_5, x_4, x_6, x_2)}. \end{aligned}$$

Eq. 71 follows. Then (72) follows from (44). □

6 Proof of Theorem 1

On the analogy of Lemma 2, it suffices to show the following:

Theorem 1'.

The two-jet $(R, \nabla R, \nabla^2 R)$ of some curvature tensor is Einstein if and only if $(R, \nabla R)$ is Einstein and

$$\begin{array}{|c|c|c|c|} \hline 1 & 3 & 5 & 6 \\ \hline 2 & 4 & & \\ \hline \end{array} (\nabla_{x_5, x_6}^2 R(x_1, x_2, x_3, x_4) - \frac{1}{n+4} \langle x_5, x_6 \rangle (R * R)(x_1, x_2, x_3, x_4)) \quad (73)$$

is totally trace-free.

Proof of Theorem 1'. In the one direction, suppose the Einstein condition holds. Clearly this implies that $R * \text{ric} = 0$. Hence $R * R$ has vanishing Ricci tensor according to (22). Thus, on the one hand applying Corollary 7 to the curvature tensor $R * R$, we obtain that

$$\begin{aligned} & -\frac{1}{n+4} \text{tr}_{1,3} \begin{array}{|c|c|c|c|} \hline 1 & 3 & 5 & 6 \\ \hline 2 & 4 & & \\ \hline \end{array} \langle x_5, x_6 \rangle R * R(x_1, x_2, x_3, x_4) \\ & \stackrel{(71)}{=} -4(R * R(x_5, x_2, x_6, x_4) + R * R(x_5, x_4, x_6, x_2)) . \end{aligned}$$

On the other hand,

$$-\text{tr}_{1,3} \begin{array}{|c|c|c|c|} \hline 1 & 3 & 5 & 6 \\ \hline 2 & 4 & & \\ \hline \end{array} \nabla_{x_5, x_6}^2 R(x_1, x_2, x_3, x_4) = -4(R * R(x_5, x_2, x_6, x_4) + R * R(x_5, x_4, x_6, x_2))$$

according to Corollary 5. Thus the trace of (73) with respect to $\{x_1, x_3\}$ vanishes. Because of Lemma 4 this already implies that (73) is totally trace-free. □

In the other direction, suppose that $(R, \nabla R)$ satisfies the Einstein condition and that (73) is totally trace-free. We aim to show that $\nabla^2 \text{ric} = 0$. For this, since the Ricci tensor is assumed to be a multiple of the identity,

$$\begin{aligned} & \nabla_{x_1, x_2}^2 \text{ric}(y_1, y_2) - \nabla_{x_2, x_1}^2 \text{ric}(y_1, y_2) , \\ & = R_{x_1, x_2} \cdot \text{ric}(y_1, y_2) = 0 . \end{aligned}$$

Thus we already know that

$$\nabla^2 \text{ric} \in \text{Sym}^2 V^* \otimes \text{Sym}^2 V^* . \quad (74)$$

Further, recall from the Littlewood-Richardson rules that there is an abstract $\mathfrak{sl}(V; \mathbb{C})$ decomposition

$$\text{Sym}^2 V^* \otimes \text{Sym}^2 V^* = \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline \end{array} \oplus \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \quad (75)$$

where each Young frame represents some irreducible component in $\bigotimes^4 V^*$. We will show that each of the three components of $\nabla^2 \text{ric}$ vanishes:

First, we aim to show that

$$\forall_{x_1, x_2, x_3, x_4 \in V} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & & \end{bmatrix} \nabla_{x_1, x_3}^{(2)} \text{ric}(x_2, x_4) = 0. \quad (76)$$

This means by the polarization formula that $\nabla_{\xi, \xi}^{(2)} \text{ric}(\xi, \xi) = 0$ for all $\xi \in V$.

For this, we calculate the trace of (73) with respect to $\{x_1, x_3\}$ and evaluate for $x_2 = x_4 = x_5 = x_6 = \xi$:

$$\begin{aligned} & \underbrace{\text{tr}_{1,3} \begin{bmatrix} 1 & 3 & 5 & 6 \\ 2 & 4 & & \end{bmatrix} \nabla_{x_5, x_6}^{(2)} R(x_1, x_2, x_3, x_4)|_{x_2=x_4=x_5=x_6=\xi}}_{\stackrel{(53)}{=} -80 \nabla_{\xi, \xi}^{(2)} \text{ric}(\xi, \xi)} \\ & \stackrel{!}{=} \frac{1}{n+4} \underbrace{\text{tr}_{1,3} \begin{bmatrix} 1 & 3 & 5 & 6 \\ 2 & 4 & & \end{bmatrix} \langle x_5, x_6 \rangle R * R(x_1, x_2, x_3, x_4)|_{x_2=x_4=x_5=x_6=\xi}}_{\stackrel{(71)}{=} 0}. \end{aligned}$$

We conclude that $\nabla_{\xi, \xi}^{(2)} \text{ric}(\xi, \xi) = 0$. □

Second, we will show that

$$\forall_{x_1, x_2, x_3, x_4 \in V} : \begin{bmatrix} 1 & 3 & 4 \\ 2 & & \end{bmatrix} \nabla_{x_1, x_2}^{(2)} \text{ric}(x_3, x_4) = \begin{bmatrix} 1 & 2 & 4 \\ 3 & & \end{bmatrix} \nabla_{x_1, x_2}^{(2)} \text{ric}(x_3, x_4) = 0. \quad (77)$$

Thus it suffices to show that $\nabla_{\xi, x}^{(2)} \text{ric}(\xi, \xi) = \nabla_{\xi, \xi}^{(2)} \text{ric}(x, \xi) = 0$ for all x, ξ :

we calculate the trace of (73) with respect to $\{x_1, x_3\}$ and evaluate for $x_2 = x_4 = x_5 = \xi$, $x_6 = x$:

$$\begin{aligned} & \underbrace{\text{tr}_{1,3} \begin{bmatrix} 1 & 3 & 5 & 6 \\ 2 & 4 & & \end{bmatrix} \nabla_{x_5, x_6}^{(2)} R(x_1, x_2, x_3, x_4)|_{x_2=x_4=x_5=\xi, x_6=x}}_{\stackrel{(53), R\text{-ric}=0}{=} 80 \nabla_{\xi, x}^{(2)} \text{ric}(\xi, \xi)} \\ & \stackrel{!}{=} \frac{1}{n+4} \underbrace{\text{tr}_{1,3} \begin{bmatrix} 1 & 3 & 5 & 6 \\ 2 & 4 & & \end{bmatrix} \langle x_5, x_6 \rangle R * R(x_1, x_2, x_3, x_4)|_{x_2=x_4=x_5=\xi, x_6=x}}_{\stackrel{(71)}{=} 0}. \end{aligned}$$

We conclude that $\nabla_{\xi, x}^{(2)} \text{ric}(\xi, \xi) = 0$. The conclusion that $\nabla_{\xi, \xi}^{(2)} \text{ric}(x, \xi) = 0$ is derived in the same way. □

Last we have to show that

$$\forall_{x_1, x_2, x_3, x_4 \in V} : \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \nabla_{x_1, x_3}^{(2)} \text{ric}(x_2, x_4) = 0. \quad (78)$$

For this, we calculate the trace of (73) with respect to $\{x_5, x_6\}$ to see that

$$\underbrace{\text{tr}_{5,6} \begin{bmatrix} 1 & 3 & 5 & 6 \\ 2 & 4 & & \end{bmatrix} \nabla_{x_5, x_6}^{(2)} R(x_1, x_2, x_3, x_4)}_{\stackrel{(63)}{=} -80 \nabla^* \nabla R(x_1, x_2, x_3, x_4) - 16 R * R(x_1, x_2, x_3, x_4)} \stackrel{!}{=} \frac{1}{n+4} \underbrace{\text{tr}_{5,6} \begin{bmatrix} 1 & 3 & 5 & 6 \\ 2 & 4 & & \end{bmatrix} \langle x_5, x_6 \rangle R * R(x_1, x_2, x_3, x_4)}_{\stackrel{(72)}{=} 24 R * R(x_1, x_2, x_3, x_4)}.$$

We conclude that $\nabla^* \nabla R + \frac{1}{2} R * R = 0$. Using the special case of the Weitzenböck formula (36), we see that (77) vanishes. □

7 Proof of Corollary 1

Let V be a pseudo-euclidean space with $n := \dim(V)$. Recall from Sec. 2 that $\mathcal{C}_k \otimes \mathbb{C}$ is an irreducible representation of $\mathfrak{sl}(V, \mathbb{C})$ of highest weight $(k+2)L_1 + 2L_2$. The subset $[k+2, 2] \subset \mathcal{C}_k$ given by the totally trace-free tensors is a representation of $\mathfrak{so}(V)$. According to [4, Thm. 19.22], the complexification $[k+2, 2] \otimes \mathbb{C}$ is

- an irreducible representation of $\mathfrak{so}(V, \mathbb{C})$ with highest weight $(k+2)L_1 + 2L_2$ (if $n \geq 5$),
- the sum of two irreducible representations of $\mathfrak{so}(V, \mathbb{C})$ with highest weights $(k+2)L_1 \pm 2L_2$ (if $n = 4$),
- or $[k+2, 2] = \{0\}$ (if $n \leq 3$).

This is called Weyls construction for orthogonal groups. Now we can give the proof of Corollary 1:

Proof. Suppose that (7) holds. By means of Lemma 2,

$$\text{tr}_{1,3} \begin{array}{|c|c|c|c|} \hline 1 & 3 & 5 & 6 \\ \hline 2 & 4 & & \\ \hline \end{array} (\nabla^2) R_{x_5, x_6}(x_1, x_2, x_3, x_4) - c \langle x_5, x_6 \rangle R(x_1, x_2, x_3, x_4) = 0. \quad (79)$$

Using additionally (73) and substituting $R * R = -2\nabla^* \nabla R$ according to (37), we thus see that

$$\text{tr}_{1,3} \begin{array}{|c|c|c|c|} \hline 1 & 3 & 5 & 6 \\ \hline 2 & 4 & & \\ \hline \end{array} \left(\frac{2}{n+4} \langle x_5, x_6 \rangle \nabla^* \nabla R_{x_5, x_6}(x_1, x_2, x_3, x_4) + c \langle x_5, x_6 \rangle R(x_1, x_2, x_3, x_4) \right) = 0. \quad (80)$$

Further, recall from (35) that on the one hand there is a decomposition into $\text{SO}(V)$ -modules

$$\mathcal{C}_0 = [0] \oplus [2] \oplus [2, 2].$$

Let $[4, 2] \subset \mathcal{C}_2$ denote the subset of totally trace-free tensors. Since the embedding ι defined in (65) is clearly $\text{SO}(V)$ -equivariant, Schurs Lemma in combination with (35) implies that $[4, 2] \cap \iota(\mathcal{C}_0) = \{0\}$. Hence $\hat{\nabla}^2 \hat{R}$ is completely determined by its traces and then by $\text{tr}_{1,3}$ as was shown in Lemma 4. We conclude from (80) that

$$\frac{2}{n+4} \nabla^* \nabla R_{x_5, x_6}(x_1, x_2, x_3, x_4) + c R(x_1, x_2, x_3, x_4) = 0. \quad (81)$$

Thus $\nabla^* \nabla R = -\frac{(n+4)c}{2} R$, which finishes the proof of (8). \square

8 Concluding remarks

Let $(V, \langle \cdot, \cdot \rangle)$ be a pseudo-euclidean vector space. It is known from the theory of partial differential equations that an algebraic k -jet is Einstein if and only if it is actually the k -jet of the curvature tensor of some Einstein metric defined in a neighborhood of the origin of V (for $k = 0$ see [6].) It follows (in complete analogy to the Jet Isomorphism Theorem) that the space of Einstein k -jets is an affine vector bundle over the space of Einstein $k-1$ -jets with direction space $[k+2, 2]$, the totally traceless part of \mathcal{C}_k .

However, to the authors best knowledge, this existence result does not give a hint how to construct the Einstein metric explicitly from the given k -jet. Now Theorem 1 points exactly into that direction. Namely, it tells us how to extend a given Einstein one-jet $(R, \nabla R)$ to an Einstein two-jet in an explicit way:

we may extend $(R, \nabla R)$ to some two-jet $(R, \nabla R, \nabla^2 R)$. In fact,

$$\text{Id} - \frac{1}{3}\mathcal{R} - \frac{1}{6}\mathcal{R}^1 : \xi \mapsto \text{Id} - \frac{1}{3}R(\cdot, \xi, \xi) - \frac{1}{6}\nabla_\xi R(\cdot, \xi, \xi) \in \text{End}_+(V)$$

is a polynomial of degree three with values in the symmetric endomorphisms of V . Using the inner product $\langle \cdot, \cdot \rangle$ in order to identify symmetric endomorphisms with symmetric bilinear forms, we obtain a metric defined in a neighbourhood of the origin which has the prescribed two-jet $(R, \nabla R)$. Next, we remove the traces of (73) via adding a suitable $\hat{\nabla}^2 \hat{R} \in \mathcal{C}_2$ (which is the analogue of finding the Weyl part of an algebraic curvature tensor.) Then $(R, \nabla R, \nabla^2 R + \frac{1}{80}\hat{\nabla}^2 \hat{R})$ is an Einstein two-jet.

It seems reasonable that similar ideas also work for higher k -jets (i.e. $k \geq 3$).

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